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# Self-Consistent Solutions of the Plasma Transport Equations in an Axisymmetric Toroidal System

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A numerical method is presented for solving a recently derived (S. P. Hirshman and S. C. Jardin, *Phys. Fluids* 22 (1979), 731) reduced set of equations describing two-dimensional transport in tokamak plasmas. The formulation exploits the different diffusion time scales by dividing each time advancement step into two parts. In the first part, the one-dimensional surface averaged partial differential equations are advanced implicitly. In the second part, the two-dimensional generalized differential equation for the toroidal flux surface velocity is inverted directly. Accurate efficient solutions are obtained with only explicit terms coupling these two steps. Solutions are presented illustrating the validity and the accuracy of this method.

## I. INTRODUCTION

Numerical solutions of the time dependent magnetohydrodynamic transport equations presently play an active role in the tokamak fusion program. Although relatively simple descriptions are often adequate, as our understanding of the microscopic processes leading to plasma transport increases, it becomes increasingly important to utilize evermore accurate and complete solution procedures to aid in exploring consequences of the theory and in interpreting experimental data.

Recently a complete formulation of two-dimensional transport in tokamak plasmas appeared in the literature [1]. This work derives a reduced set of transport equations using an expansion based only the smallness of the Alfvén transit time compared to the resistive diffusion time, and on the smallness of the perpendicular mobilities and thermal conductivities compared to their parallel values. These orderings justify using an asymptotic theory which neglects plasma inertia and which does not allow temperature variation along the magnetic field lines. The reduced equations are naturally expressed in a magnetic flux coordinate system which evolves in time self-consistently as the plasma and fields evolve..

In the present article we describe a numerical procedure for solving the reduced set of transport equations and illustrate its usefulness and validity by presenting some selected applications. In the following section we summarize the relevant equations.



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These consist primarily of five one-dimensional time advancement equations for the surface averaged field and plasma thermodynamic quantities, and a single two-dimensional linear generalized differential equation for the velocity of the toroidal magnetic flux surfaces.

Section III describes an implicit finite difference method for solution of the one-dimensional surface averaged equations. The method is general in that it is applicable to an arbitrary transport model. Section IV describes a direct method, based on Fourier series, for inverting the two-dimensional generalized differential equation. This is also valid for an arbitrary transport model.

Section V presents an approximate analytic solution, based on the smallness of the inverse aspect ratio, of the full set of self-consistent two-dimensional transport equations for the special case of FCT heating. This has been included to illustrate the relative magnitude and the causal relations between the terms in the transport and equilibrium evolution equations for standard tokamak parameters. In Section VI we present results modeling a realistic tokamak discharge with particle and energy sources and with large geometry changes.

## II. The Plasma Transport Equations

The reduced set of transport equations, derived in Ref. [1], can be summarized as follows:

$$(N_j')_t + (N_j' u)_\psi = -(V'\Gamma_j)_\psi + V'\langle S_{nj}\rangle, \tag{1a}$$

$$\sigma_t' + (\sigma' u)_\psi = -\frac{2}{5}\left(\frac{\sigma'}{p}\right) S, \tag{1b}$$

$$(\sigma_e')_t + (\sigma_e' u)_\psi = -\frac{2}{5}\left(\frac{\sigma_e'}{p_e}\right) S_e, \tag{1c}$$

$$\chi_t' + (\chi' u)_\psi = 2\pi(E_\parallel^*)_\psi, \tag{1d}$$

$$\Psi_t' + (\Psi' u)_\psi = 0, \tag{1e}$$

$$[(16\pi^3 x^2)^{-1}\chi' \Delta^* + L_0 + L_1]\,\Omega = 2\mathbf{J}\cdot\nabla\phi(E_\parallel^*)_\psi + \tfrac{1}{2}B_p^2[(E_\parallel^*)_\psi/\chi']_\psi - \tfrac{2}{3}S_\psi, \tag{2}$$

$$x_t = \psi^{-l}(\xi_\theta x_\psi - \xi_\psi x_\theta), \tag{3a}$$

$$z_t = \psi^{-l}(\xi_\theta z_\psi - \xi_\psi z_\theta). \tag{3b}$$

Here the surface averaged entropy and electron entropy source terms are defined as

$$S \equiv -\langle \mathbf{J}\cdot\nabla\phi\rangle E_\parallel^* + \frac{1}{V'}\sum_{j=i,e}[Q_j] - \langle S_p\rangle, \tag{4a}$$

$$S_e \equiv -\langle \mathbf{J}\cdot\nabla\phi\rangle E_\parallel^* + \frac{1}{V'}[Q_e]_\psi + (p_i)_\psi \Gamma_i/n_i + \langle \mathbf{u}_i\cdot\nabla\cdot\boldsymbol{\pi}_i\rangle - Q_{\Delta e} - \langle S_{pe}\rangle, \tag{4b}$$

and the surface averaged cross field heat flux term for species $j$ is

$$Q_j \equiv V'(\langle \mathbf{q}_j \cdot \nabla\psi \rangle + \tfrac{5}{2}\Gamma_j T_j). \tag{5}$$

The flux coordinates $(\psi, \theta)$ are the independent variables, along with the time $t$. Using any of these as a subscript denotes differentiation with the other two held fixed. The coordinates $(x, \phi, z)$ form a cylindrical coordinate system with $\phi$ being the symmetry angle. Both $x$ and $z$ are considered to be functions of $\psi$, $\theta$, and $t$. The Jacobian of the transformation from $\mathbf{x}$ to $(\psi, \theta, \phi)$ is defined as $J = [\nabla\psi \times \nabla\theta \cdot \nabla\phi]^{-1}$. It is taken to be of the form

$$J = Tx^m \psi^l \tag{6}$$

where $T$ is a normalization constant and $m$ and $l$ are integers. (Note that the symbol $l$ is replacing the symbol $n$ used in Ref. [1].) If the Jacobian is initially of the form (6) and if $x$ and $z$ are advanced using Eq. (3), then $J$ will always have the form of Eq. (6).

Angular brackets denote the flux surface average operator,

$$\langle a \rangle \equiv \int_0^{2\pi} d\theta\, Ja \bigg/ \int_0^{2\pi} d\theta\, J, \tag{7}$$

and the differential volume is defined as

$$V'(\psi, t) \equiv \partial\left(\int_0^{\psi} d\mathbf{x}\right) \bigg/ \partial\psi = 2\pi \int_0^{2\pi} J\, d\theta. \tag{8}$$

The variables advanced in time in Eqs. (1a) through (1e) are the differential particle number for each species $N_j'(\psi, t) \equiv n_j(\psi, t)\, V'$, the differential plasma entropy $\sigma'(\psi, t) \equiv [p(\psi, t)]^{3/5}\, V'$, the differential electron entropy $\sigma_e'(\psi, t) \equiv [p_e(\psi, t)]^{3/5}\, V'$, the poloidal flux density $\chi'(\psi, t) \equiv (2\pi)^{-1}\, \partial(\int_0^{\psi} d\mathbf{x}\, \mathbf{B} \cdot \nabla\theta)/\partial\psi = 2\pi J\mathbf{B} \cdot \nabla\theta$, and the toroidal flux density $\Psi'(\psi, t) \equiv (2\pi)^{-1}\, \partial(\int_0^{\psi} d\mathbf{x}\, \mathbf{B} \cdot \nabla\phi)/\partial\psi = (2\pi)^{-1}\, g\langle x^{-2}\rangle\, V'$. Here $n_j$ is the particle density of species $j$, $p_j$ is the pressure of species $j$, $p = p_e + p_i$, and the magnetic field is $\mathbf{B} = \mathbf{B}_p + \mathbf{B}_T$, with $\mathbf{B}_p = (2\pi)^{-1}\chi'(\psi, t)\, \nabla\phi \times \nabla\psi$ and $\mathbf{B}_t = g(\psi, t)\, \nabla\phi$. These are related by the Grad–Shafranov equilibrium equation

$$\Delta^*\chi + (2\pi)^2(gg_\psi + 4\pi x^2 p_\psi)(\chi')^{-1} = 0. \tag{9}$$

The operators $\Delta^*$, $L_0$, and $L_1$ in Eq. (2) are defined by

$$\Delta^* F \equiv x^2 J^{-1}[(h^{\psi\psi}F_\psi + h^{\psi\theta}F_\theta)_\psi + (h^{\psi\theta}F_\psi + h^{\theta\theta}F_\theta)_\theta], \tag{10}$$

where the $h^{\alpha\beta} \equiv x^{-2} J \nabla\alpha \cdot \nabla\beta$ can be expressed as derivatives of the cylindrical coordinates as $h^{\psi\psi} = J^{-1}(x_\theta^2 + z_\theta^2)$, $h^{\theta\theta} = J^{-1}(x_\psi^2 + z_\psi^2)$, $h^{\psi\theta} = -J^{-1}(x_\theta x_\psi + z_\theta z_\psi)$ and

$$L_0(a) = (4\pi x^2)^{-1}\{g^2(V'\langle x^{-2}\rangle)^{-1}[(V'/\chi')\langle x^{-2}a\rangle]_\psi\}_\psi, \tag{11a}$$

$$L_1(a) = [(p_\psi/\chi')_\psi + (4\pi x^2)^{-1}(gg_\psi/\chi')_\psi]\, a + 5/3\{(p/V')[(V'/\chi')\langle a\rangle]_\psi\}_\psi. \tag{11b}$$

Note that

$$\mathbf{J} \cdot \nabla\phi = (8\pi J)^{-1}[(h^{\psi\psi}\chi_\psi)_\psi + (h^{\psi\theta}\chi_\psi)_\theta] \tag{12a}$$

$$= -2\pi[(4\pi x^2)^{-1} gg_\psi + p_\psi](\chi')^{-1}, \tag{12b}$$

where the second form is obtained using Eq. (9).

We have used the symbol $\Omega$, Eq. (2), to denote $(\chi'/\Psi')$ times the absolute time derivative of the toroidal flux function, or equivalently, $\chi'$ times the normal component of the toroidal flux velocity. Thus

$$\Omega \equiv q^{-1}(d\Psi/dt)\,|_{\mathbf{x}} = \chi' \mathbf{u}_\Psi \cdot \nabla\psi,$$

where $q \equiv \Psi'/\chi'$ is the tokamak safety factor. As derived in Ref. [1], the relative velocity of $\Psi$ and $\psi$ surfaces, $u(\psi,t)$, and the derivatives of the coordinate stream function, $\xi_\theta(\psi, \theta, t)$ and $\xi_\psi(\psi, \theta, t)$, are obtainable from $\Omega$ of Eq. (2) by the relations

$$u(\psi, t) = \langle x^{-m}\Omega\rangle/(\langle x^{-m}\rangle\, \chi'), \tag{13a}$$

$$\xi_\theta(\psi, \theta, t) = (\Omega - \langle x^{-m}\Omega\rangle/\langle x^{-m}\rangle)(\psi^l/\chi'), \tag{13b}$$

$$\xi_\psi(\psi, \theta, t) = \partial/\partial\psi \int_0^\theta \xi_\theta\, d\theta. \tag{13c}$$

Physically, the formulation of the two-dimensional transport problem according to Eqs. (1) through (3) amounts to a mixed Lagrangian–Eulerian description. The one-dimensional Eqs. (1a) through (1e) are Eulerian in nature since the time derivatives are taken with the purely geometrical coordinate

$$\psi = \left\{[(l+1)/2\pi T] \int_0^V \left[\int_0^{2\pi} x^m\, d\theta\right]^{-1} dV\right\}^{1/(l+1)}$$

held fixed. Each of these equations contains the convective velocity $u$ which describes how suraces of constant toroidal flux convect relative to the constant $\psi$ surfaces. The terms on the right-hand side of Eq. (1) describe source terms and transport effects which cause slippage between the other adiabatic variables and the constant toroidal flux surfaces.

The description is partly Lagrangian in that the $(x, z)$ coordinates of the $(\psi, \theta)$ grid evolve in time with the velocity $\mathbf{v}_g = x^m \nabla \times (\xi\nabla\phi)$ according to Eq. (3). Thus the two-dimensional variable $\Omega(\psi, \theta)$, which is obtained from Eq. (2) and describes how the toroidal flux surfaces change in time, gives two pieces of information. Its flux surface average, Eq. (13a), gives a one-dimensional velocity $u$ with respect to an Eulerian grid equally spaced in $\psi$. Its surface varying part, Eq. (13b), gives the Lagrangian coordinate stream function $\xi$ which describes how the constant $\psi$ surfaces change their shape.

Once the transport quantities $E_{\parallel}^{*}$, $\Gamma_j$, $Q_j$ and $\langle \mathbf{u}_i \cdot \nabla \cdot \boldsymbol{\pi}_i \rangle$ are given in terms of the plasma variables and their derivatives, the system of Eqs. (1) through (3) together with the definitions in Eqs. (4) through (13) provide a closed system, needing only the source functions $\langle S_{nj} \rangle$, $\langle S_p \rangle$, $\langle S_{pe} \rangle$, and the boundary conditions to completely specify a problem. The numerical method used to solve Eqs. (1) through (3) is outlined here and detailed in the next two sections.

If we denote by $\mathbf{Y}$ the vector consisting of $N_j'$, $\sigma'$, $\sigma_e'$, $\chi'$, and $\Psi'$, then Eqs. (1) through (3) can be written symbolically as

$$\mathbf{x}_t = x^m \nabla \times (\xi \nabla \phi), \tag{3'}$$

$$\mathbf{Y}_t + (\mathbf{Y}u)_{\psi} = \mathbf{A}(\mathbf{Y}, \boldsymbol{x}), \tag{1'}$$

$$L(\mathbf{Y}, \mathbf{x})\{\Omega\} = \mathbf{B}(\mathbf{Y}, \mathbf{x}). \tag{2'}$$

Here, of course, $u$, $\xi$, and $\Omega$ are related by Eq. (13) which implies $\Omega = \chi'(\psi^{-l}\xi_\theta + u)$. The notation used in Eq. (2)′ means that the operator $L$ is an explicit function of $\mathbf{Y}$ and $\mathbf{x}$ and acts on the function $\Omega$.

As discussed in Ref. [1], the formulation adopted here ensures that the toroidal flux velocity $u$ is "small" compared to the other diffusive velocities; i.e., $u \sim B_p^2/B_T^2 \ll 1$. This justifies adopting the following prescription for advancing the solution from time level $n$ to time level $(n+1)$:

$$[\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}]/\Delta t = x^m \nabla \times [\xi^{(n)} \nabla \phi], \tag{3''}$$

$$[\mathbf{Y}^{(n+1)} - \mathbf{Y}^{(n)}]/\Delta t + [\mathbf{Y}^{(n+1/2)} u^{(n)}]_{\psi} = \mathbf{A}[\mathbf{Y}^{(n+1/2)}, \mathbf{x}^{(n+1/2)}], \tag{1''}$$

$$L[\mathbf{Y}^{(n+1)}, \mathbf{x}^{(n+1)}]\{\Omega^{(n+1)}\} = \mathbf{B}[\mathbf{Y}^{(n+1)}, \mathbf{x}^{(n+1)}]. \tag{2''}$$

The spatial finite difference method used in Eq. (3)″ involves only straightforward centered differences and its description will be omitted here. Since only the old time level of $u$, $u^{(n)}$, appears in Eq. (1)″, Eqs. (1)″ and (2)″ decouple and their finite difference solution can be considered separately. This is the subject of the following two sections.

## III. A Numerical Method for the One-Dimensional System of Surface Averaged Transport Equations

Here we detail a numerical method for advancing in time the one-dimensional quantities $N'(\psi, t)$, $\sigma'(\psi, t)$, $\sigma_e'(\psi, t)$, $\chi'(\psi, t)$ according to Eqs. (1a) through (1d). Since the right-hand side of Eq. (1e) is zero, the finite difference equation for $\Psi'$ splits off from the others and is solved simply by applying centered spatial difference operators. As discussed in Section II, the relative toroidal flux velocity $u$ and the geometrical quantities $V'$, $\langle |\nabla\psi|^2/x^2\rangle$, $\langle x^{-2}\rangle$ are taken to be known here.

There are five transport model dependent quantities describing relative slippage in a two-fluid resistive high temperature plasma; the parallel electric field $\alpha \equiv 2\pi E_\parallel^* = 2\pi\langle \mathbf{E}\cdot\mathbf{B}\rangle/\langle \mathbf{B}\cdot\nabla\phi\rangle$, the relative particle flux $\beta = \Gamma_i/n_i = \Gamma_e/n_e$, the ion heat flux $\gamma \equiv Q_i = V'(\langle \mathbf{q}_i\cdot\nabla\psi\rangle + 5/2\Gamma_i T_i)$, the electron heat flux $\delta \equiv Q_e = V'(\langle \mathbf{q}_e\cdot\nabla\psi\rangle + 5/2\Gamma_e T_e)$, and the viscous heating term $\varepsilon \equiv V'\langle \mathbf{u}_i\cdot\nabla\cdot\boldsymbol{\pi}_i\rangle$. It is the objective of transport theory to relate these fluxes to the vector of forces consisting of derivatives of the fundamental thermodynamic variables. It is convenient to define the force vector as

$$\mathbf{F} \equiv [1, (N'/V')_\psi, (\sigma'/V')_\psi, (\sigma_e'/V')_\psi, (\chi' V'\langle|\nabla\psi|^2/x^2\rangle/g)_\psi]. \tag{14}$$

To specify a particular transport model, it is then sufficient to supply the 25 functions $(\alpha^j, \beta^j, \gamma^j, \delta^j, \varepsilon^j)$, $j = 0, 4$, according to the format

$$\alpha = \sum_{j=0}^{4} \alpha^j F^j, \qquad \beta = \sum_{j=0}^{4} \beta^j F^j, \qquad \text{etc.} \tag{15}$$

We introduce the notation that a subscript $j$ corresponds to location $\psi = [j - (1/2)]\,\Delta\psi$ and a superscript $n$ corresponds to time level $t = (n-1)\,\Delta t$, and finite difference according to the Crank–Nicholson method with $\theta$ being the implicitness parameter. A value of $\theta = 1$ corresponds to the method being fully implicit; i.e., the highest spatial derivatives are evaluated at the advanced time level $(n+1)$. The finite difference form of the surface averaged transport equations, Eqs. (1a) through (1d), can then be written in the form

$$\mathbf{A}_j^n \cdot \mathbf{\Phi}_{j+1}^{n+1} - \mathbf{B}_j^n \cdot \mathbf{\Phi}_j^{n+1} + \mathbf{C}_j^n \cdot \mathbf{\Phi}_{j-1}^{n+1} + \mathbf{D}_j^n = 0. \tag{16}$$

Here $\mathbf{\Phi}_j^n$ is the solution vector at position $j$ and time level $n$,

$$\mathbf{\Phi}_j^n \equiv [N_j'^n, \sigma_j'^n, \sigma_{ej}'^n, \chi_j'^n],$$

and the elements of the matrices $\mathbf{A}_j^n$, $\mathbf{B}_j^n$, $\mathbf{C}_j^n$ and the vectors $\mathbf{D}_j^n$ are given in terms of the $\alpha^j, \beta^j$, etc., in Appendix A.

The matrix tridiagonal system, Eq. (16), is inverted the usual way [2] with fixed value or derivative boundary conditions easily being incorporated at the boundary $j = J$. Thus, the matrices $\mathbf{E}_j$ and the vectors $\mathbf{F}_j$ are defined by the recursion relations

$$\mathbf{E}_1 = 0, \tag{17a}$$

$$\mathbf{F}_1 = 0, \tag{17b}$$

$$\mathbf{E}_j = (\mathbf{B}_j - \mathbf{C}_j \cdot \mathbf{E}_{j-1})^{-1} \cdot \mathbf{A}_j; \qquad j > 1, \tag{17c}$$

$$\mathbf{F}_j = (\mathbf{B}_j - \mathbf{C}_j \cdot \mathbf{E}_{j-1})^{-1} \cdot (\mathbf{D}_j + \mathbf{C}_j \cdot \mathbf{F}_{j-1}); \qquad j > 1. \tag{17d}$$

The outer boundary condition is expressed in the form

$$-\mathbf{B}_J \cdot \mathbf{\Phi}_{J-1}^{n+1} + \mathbf{C}_J \cdot \mathbf{\Phi}_{J-1}^{n+1} + \mathbf{D}_J = 0. \tag{18}$$

Then

$$\mathbf{\Phi}_{J-1}^{n+1} = (\mathbf{I} - \mathbf{E}_{J-1} \cdot \mathbf{B}_J^{-1} \cdot \mathbf{C}_J)^{-1} \cdot (\mathbf{E}_{J-1} \cdot \mathbf{B}_J^{-1} \cdot \mathbf{D}_J + \mathbf{F}_{J-1}), \tag{19a}$$

$$\mathbf{\Phi}_J^{n+1} = \mathbf{B}_J^{-1} \cdot [\mathbf{C}_J \cdot \mathbf{\Phi}_{J-1}^{n+1} + \mathbf{D}_J], \tag{19b}$$

and the interior $\mathbf{\Phi}_j^{n+1}$ are obtained from the recursion relations

$$\mathbf{\Phi}_j^{n+1} = \mathbf{E}_j \cdot \mathbf{\Phi}_{j+1}^{n+1} + \mathbf{F}_j. \tag{20}$$

Advancing the solution vector $\mathbf{\Phi}$ according to Eqs. (16) through (20) and using an implicitness parameter $\theta \geqslant \frac{1}{2}$ eliminates a maximum time step criterion based on numerical stability. Off diagonal terms are treated on an equal footing with diagonal terms. Because of the nonlinear nature of the problem (the $\mathbf{\Phi}$ dependence of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$) we impose the accuracy restriction that no component of $\Phi$ change by more than 5% at any grid point at any time step. If this criterion is violated, the time step is repeated. In practice we find this time advancement method to be extremely rugged, allowing time steps several hundred times the explicit time step criteria when the solution is approaching a steady state.

## IV. A Direct Solver for a Two-Dimensional Non-local Linear Generalized Differential Equation

We describe here a numerical method for the direct solution of equations of the type

$$(\Omega_\theta a + \Omega_\psi b)_\psi + (\Omega_\theta c + \Omega_\psi a)_\theta + d\Omega + JH[P(V'\langle H\Omega\rangle)_\psi]_\psi + JK[G(V'\langle K\Omega\rangle)_\psi]_\psi + e = 0. \tag{21}$$

Subscripts here denote differentiation and brackets denote the usual flux surface average operator, defined in Eq. (7). Equation (21) is equivalent to Eq. (2) if we make the identification

$$a(\psi, \theta) = h^{\psi\theta} = -J^{-1}(x_\theta x_\psi + z_\theta z_\psi), \tag{22a}$$

$$b(\psi, \theta) = h^{\psi\psi} = J^{-1}(x_\theta^2 + z_\theta^2), \tag{22b}$$

$$c(\psi, \theta) = h^{\theta\theta} = J^{-1}(x_\psi^2 + z_\psi^2), \tag{22c}$$

$$d(\psi, \theta) = \frac{(2\pi)^2 J}{\chi'} [4\pi(p_\psi/\chi')_\psi + x^{-2}(gg_\psi/\chi')_\psi], \tag{22d}$$

$$e(\psi, \theta) = \frac{-16\pi^3 J}{\chi'} \left\{ 2\mathbf{J} \cdot \nabla\phi (E_\parallel^*)_\psi + \frac{1}{2} B_p^2 [(E_\parallel^*)_\psi / \chi']_\psi - \frac{2}{3} S_\psi \right\}, \tag{22e}$$

$$K(\psi, \theta) = x^{-2}/\chi', \tag{22f}$$

$$H(\psi, \theta) = 1/\chi', \tag{22g}$$

$$G(\psi) = (2\pi)^2 g^2/(V' \langle x^{-2} \rangle), \tag{22h}$$

$$P(\psi) = \frac{5}{3} 16\pi^3 p/V'. \tag{22i}$$

Equation (21) is a linear non-local two-dimensional equation for $\Omega$ to be inverted each time step determining the time evolution of the toroidal flux surfaces through the velocity variables $u$ and $\xi$, Eq. (13). Mathematically, it is a generalized differential equation [1] requiring boundary conditions for $\Omega(\psi_{\max}, \theta)$ and a single boundary condition for $\Omega(0, \theta)$. The $\theta$ integrals (or flux surface averages) in Eq. (21) make it non-local and thus impractical to solve by iteration techniques. Instead we solve this equation directly by Fourier decomposing in the angle coordinate $\theta$ and by finite differencing in the surface coordinate $\psi$. Fourier techniques are natural in that they treat $\theta$ integrals and $\theta$ derivatives on an equal footing. The efficacy of this method lies in the fact that for most problems, only 5–10 Fourier harmonics need be kept at each $\psi$ surface to obtain accurate solutions.

For simplicity we restrict discussion to equations which are symmetric about the plasma midplane. Then $\Omega$ has a discrete Fourier representation of the form $\Omega(\psi, \theta) = \Omega^0(\psi) + \sum_{m=1}^{M} \Omega^m(\psi) \cos m\theta$, and the coefficients can be expanded

$$\begin{bmatrix} b(\psi,\theta) \\ c(\psi,\theta) \\ d(\psi,\theta) \\ e(\psi,\theta) \\ K(\psi,\theta) \\ H(\psi,\theta) \end{bmatrix} = 1/2 \begin{bmatrix} b^0(\psi) \\ c^0(\psi) \\ d^0(\psi) \\ e^0(\psi) \\ K^0(\psi) \\ H^0(\psi) \end{bmatrix} + \sum_{m=1}^{M} \begin{bmatrix} b^m(\psi) \\ c^m(\psi) \\ d^m(\psi) \\ e^m(\psi) \\ K^m(\psi) \\ H^m(\psi) \end{bmatrix} \cos m\theta \tag{23}$$

and

$$a = \sum_{m=1}^{M} a^m(\psi) \sin m\theta.$$

By substituting Eq. (23) into Eq. (21) evaluating the product terms, and finite differencing in $\psi$ with the notation $\psi_j = j\,\Delta\psi$, we obtain the canonical system

$$\mathbf{A}_j \cdot \mathbf{\Omega}_{j+1} - \mathbf{B}_j \cdot \mathbf{\Omega}_j + \mathbf{C}_j \cdot \mathbf{\Omega}_{j-1} + \mathbf{D}_j = 0. \tag{24}$$

Here $\mathbf{\Omega}_j \equiv \{\Omega_j^0, \Omega_j^1, \ldots, \Omega_j^M\}$ is a vector containing the coefficients of the harmonics of $\mathbf{\Omega}$ at location $\psi_j$. The elements of the $(M+1) \times (M+1)$ matrices $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$, and the length $(M+1)$ vectors $\mathbf{D}_j$ are defined in Appendix B.

The system of equations (24), identical in form to Eq. (16), is solved according to the same algorithm as that given in Section III. We note that a single boundary value, for $\Omega_0^0$, need be supplied at the origin. As discussed in Ref. [1], setting $\Omega_0^0 = 0$ corresponds to no toroidal flux being created there.

Boundary values for all the $(\Omega_J^m; m = 0, M)$ must be supplied at the outer boundary, $j = J$. These either are prescribed or are calculated by Green's function techniques to be consistent with currents in external coils. Numerical implementation of the latter method will be described in a future publication.

As an initial condition, the equilibrium (Grad–Shafranov) equation, Eq. (9), is solved at time $t = 0$ using a flux coordinate equilibrium code [3]. In subsequent time steps it is in general not necessary to solve the equilibrium equation since the solution of Eq. (21) describes how the equilibrium flux surfaces change from one time step to the next. Some truncation error will accumulate in this process so that, as time progresses, the equilibrium equation will not be exactly satisfied. One can constantly monitor this error, and when it builds up to a noticeable value, the flux coordinate equilibrium code can be called in and run in an adiabatic mode to reduce this error to zero. In practice, this readjustment of the equilibrium is seldom, if ever, needed to maintain the accumulated error in the equilibrium equation to a few percent over the course of a complete problem.

## V. Flux Conserving Tokamak Solutions

Recently much interest has been shown in the concept of flux conserving tokamaks (FCT's) [4]. This refers to a situation where rapid heating is applied to a tokamak on a time scale short compared to the resistive diffusion time so that dissipation can be neglected. In this section we derive the lowest order asymptotic solutions to Eqs. (1) through (3) valid for FCT heating of a large aspect ratio ($a/R \sim \varepsilon \ll 1$), circular cross section, low beta ($8\pi p/B^2 \sim \varepsilon^2$) tokamak plasma. This approximate solution is then compared to the full numerical solution, obtained by the method described in Sections II through IV.

We specialize Eq. (6) to the case where $m = 2$ and $l = 0$; i.e., $J \sim x^2$. The coordinate functions $x(\psi, \theta, t)$ and $z(\psi, \theta, t)$ must therefore satisfy the constraint

$$J \equiv x(x_\psi z_\theta - x_\theta z_\psi) = a^2 x^2/2R. \tag{25}$$

Here the normalization has been chosen such that the plasma boundary $\psi = 1$ corresponds to a torus with minor radius $a$ and a major radius $R$. Equation (25) implies $J \sim \psi \sim \varepsilon^2$, where the tag $\varepsilon$ indicates the order in an inverse aspect ratio expansion. Equation (25) will be satisfied to orders $\varepsilon^0$ through $\varepsilon^2$ if $x(\psi, \theta, t)$ and $z(\psi, \theta, t)$ are of the form [5]

$$x(\psi, \theta, t) = R + \varepsilon a\sqrt{\psi}\cos\theta - \varepsilon^2 \Delta + \varepsilon^2 \psi(\Delta_\psi + a^2/2R)[\cos 2\theta - 1] + \cdots, \tag{26a}$$

$$z(\psi, \theta, t) = \varepsilon a\sqrt{\psi}\sin\theta + \varepsilon^2 \psi(\Delta_\psi + a^2/2R)\sin 2\theta + \cdots, \tag{26b}$$

where the shift $\Delta(\psi, t)$ remains to be determined. Using Eq. (26) we can calculate the metric quantities

$$h^{\psi\psi} = \frac{2\psi\varepsilon^2}{R}\left(1 + \frac{4\varepsilon}{a}\sqrt{\psi}\,\Delta_\psi \cos\theta + \cdots\right), \tag{27a}$$

$$h^{\theta\theta} = \frac{\varepsilon^{-2}}{2\psi R}\left[1 - \frac{4\varepsilon}{a}\left(\Delta_\psi + \frac{a^2}{2R}\right)\sqrt{\psi}\cos\theta + \cdots\right], \tag{27b}$$

$$h^{\psi\theta} = -\frac{2\varepsilon}{Ra}\sqrt{\psi}\sin\theta\left[2(\psi\Delta_\psi)_\psi + \frac{a^2}{2R} + \cdots\right], \tag{27c}$$

$$x^{-2} = R^{-2}\left(1 - 2\frac{\varepsilon a}{R}\sqrt{\psi}\cos\theta + \cdots\right), \tag{27d}$$

$$V' = 2\pi^2 R a^2 + \cdots. \tag{27e}$$

We take the standard, low beta, tokamak ordering whereby the pressure, toroidal field function, poloidal, and toroidal fluxes are ordered

$$p(\psi, t) = \varepsilon^2 p^{(2)}(\psi, t) + \cdots, \tag{28a}$$

$$g(\psi, t) = R[1 + \varepsilon^2 g^{(2)}(\psi, t) + \cdots], \tag{28b}$$

$$\chi'(\psi, t) = \chi'^{(0)}(\psi, t) + \cdots, \tag{28c}$$

$$\Psi'(\psi, t) = 2\pi R[1 + \varepsilon^2 \Psi'^{(2)}(\psi, t) + \cdots], \tag{28d}$$

As initial conditions, we choose $p^{(2)}$ to be linear in $\psi$; i.e.,

$$p^{(2)}(\psi, 0) = (8\pi)^{-1}\beta_0(1 - \psi), \tag{29a}$$

$$\sigma'(\psi, 0) = 2\pi^2 R a^2[\varepsilon^2 p^{(2)}]^{3/5}, \tag{29b}$$

and we choose the initial poloidal flux function $\chi'^{(0)}$ so that the safety factor $q$ is independent of $\psi$ to lowest order,

$$\chi'^{(0)}(\psi, 0) = \pi a^2/q_0. \tag{30}$$

The two lowest order terms in the expansion of Eq. (9), the Grad–Shafranov equilibrium equation,

$$(\pi R a)^{-2}\chi'^{(0)}[\chi'^{(0)}\psi]_\psi + g_\psi^{(2)} + 4\pi p_\psi^{(2)} = 0, \tag{31a}$$

$$(2\pi)^{-2} a^{-3}\chi'^{(0)}\left\{\psi^{-1/2}\left[\chi'^{(0)}\frac{8\psi^{3/2}}{aR}\Delta_\psi\right]_\psi - \frac{2\chi'^{(0)}}{Ra}\left[2(\psi\Delta_\psi)_\psi + \frac{a^2}{2R}\right]\right\} + 4\pi p_\psi^{(2)} = 0, \tag{31b}$$

can be integrated, using Eq. (30), to give the initial conditions for $g^{(2)}$ and for $\Delta$,

$$g^{(2)}(\psi, 0) = (\beta_0/2)(1 - \beta_\theta^{-1})(\psi - 1), \tag{32a}$$

$$\Psi'^{(2)}(\psi, 0) = g^{(2)}, \tag{32b}$$

$$\Delta(\psi, 0) = (a^2/8R)(1 + 4\beta_\theta)(\psi - 1), \tag{32c}$$

where $\beta_\theta \equiv q_0^2 R^2 \beta_0/(2a^2)$.

We now consider solving Eqs. (1) through (3) subject to the initial conditions given in Eqs. (29), (30), and (32). For FCT solutions we take the transport terms to vanish; i.e., $E_\parallel^* = \Gamma = \langle \mathbf{q}_e \cdot \nabla\psi \rangle = \langle \boldsymbol{q}_i \cdot \nabla\psi \rangle = \langle \boldsymbol{u}_i \cdot \nabla \boldsymbol{\pi}_i \rangle = 0$, and the source terms are given by $\langle S_n \rangle = \langle S_{pe} \rangle = 0$ and $\langle S_p \rangle(\psi, t) = \varepsilon^2 (3/2)(8\pi)^{-1} \dot{\beta}(1 - \psi)$.

The density and electron pressure variables, $N'$ and $\sigma_e'$, can be computed from Eqs. (1a) and (1c), but need not be since they decouple from the remainder of the solution. We anticipate, and will show below, that $u$ in Eqs. (1a) through (1e) is small, $u \sim \varepsilon^2$ so that Eqs. (1b) and (1d) can be easily integrated to give to lowest order in $\varepsilon$

$$p^{(2)}(\psi, t) = (8\pi)^{-1} \beta(t)(1 - \psi), \tag{33a}$$

$$\chi'^{(0)}(\psi, t) = \pi a^2/q_0, \tag{33b}$$

with $\beta(t) = \beta_0 + \int_0^t \dot{\beta}(t)\, dt$.

Expanding in powers of $\varepsilon$, Eq. (2) takes the form

$$\{L^{(0)} + \varepsilon L^{(1)} + \varepsilon^2 L^{(2)} + \cdots\}\{\varepsilon\Omega^{(1)} + \varepsilon^2 \Omega^{(2)} + \cdots\} = -\varepsilon^2 (8\pi)^{-1} \dot{\beta}. \tag{34a}$$

Here, the operators $L^{(0)}$, $L^{(1)}$, and $L^{(2)}$ are defined by

$$L^{(0)}\{\Omega\} \equiv \frac{q_0}{(2\pi a)^2} \left[\int_0^{2\pi} \frac{d\theta}{2\pi} \Omega\right]_{\psi\psi}, \tag{34b}$$

$$L^{(1)}\{\Omega\} \equiv -2 \frac{a}{R} \sqrt{\psi} \cos\theta\, L^{(0)}\{\Omega\}, \tag{34c}$$

$$\begin{aligned} L(2)\{\Omega\} \equiv{}& (8\pi^2 R q_0)^{-1} \left[\left(\frac{2\psi}{R} \Omega_\psi\right)_\psi + \frac{1}{2\psi R} \Omega_{\theta\theta}\right] \\ &+ 5/3 \frac{q_0}{\pi a^2} \left[p^{(2)} \left(\int_0^{2\pi} \frac{d\theta}{2\pi} \Omega\right)_\psi\right]_\psi \\ &+ \frac{q_0}{(2\pi a)^2} \left[2g^{(2)} \left(\int_0^{2\pi} \frac{d\theta}{2\pi} \Omega\right)_\psi\right]_\psi + \left(\frac{x_0^2}{x^2}\right)^{(2)} L^{(0)}\{\Omega\}. \end{aligned} \tag{34d}$$

We solve Eq. (34a) order by order with the boundary condition $\Omega = \Omega_b$ at $\psi = 1$. The introduction of the constant $\Omega_b$ allows for the possibility of toroidal flux crossing the plasma boundary. To order $\varepsilon$, Eq. (34a) gives the condition

$$\int_0^{2\pi} d\theta\, \Omega^{(1)} = 0, \tag{35}$$

i.e., to lowest order $\Omega$ has no $\theta$-independent part. This arises mathematically from the dominance of the operator $L_0$ in Eq. (2), and physically from the fact that the large externally generated toroidal field is nearly incompressible

Using the result, Eq. (35), we can solve Eq. (34) to order $\varepsilon^2$ giving

$$\Omega^{(2)} = -\frac{\dot{\beta}\pi a^2}{4q_0}\psi(\psi-1) + \Omega_b \psi. \tag{36}$$

Similarly, using both Eqs. (35) and (36), Eq. (34) can be solved to order $\varepsilon^3$ to give the equation

$$[2\psi\Omega_\psi^{(1)}]_\psi + \frac{1}{2\psi}\Omega_{\theta\theta}^{(1)} = -2\pi a\dot{\beta}q_0\sqrt{\psi}\cos\theta,$$

with solution

$$\Omega^{(1)} = -\frac{\pi}{2}aRq_0\dot{\beta}(t)(\psi^{3/2} - \psi^{1/2})\cos\theta. \tag{37}$$

From the explicit solution procedure presented here, we can anticipate the convergence problems one would encounter in attempting to solve Eq. (2) by iterative methods rather than using the direct matrix method outlined in Section IV. The largest piece of the operator in Eq. (2), $L_0$, is nonlocal. Its effect is to force the $\theta$-independent part of $\Omega$ to be small. This small $\theta$-independent part of $\Omega$, which can be obtained by flux surface averaging Eq. (2) over the angle $\theta$ then determines the larger $\theta$-dependent part.

Substitution of the results given in Eqs. (36) and (37) into Eq. (13) gives explicit expressions for the lowest order Eulerian toroidal flux velocity $u$ and for the Lagrangian coordinate stream function $\xi$,

$$u^{(2)}(\psi, t) = -1/4\dot{\beta}(t)\,\psi(\psi-1) + \frac{q_0\psi}{\pi a^2}\Omega_b, \tag{38}$$

$$\xi_\theta^{(1)}(\psi,\theta,t) = -\frac{R}{2a}q_0^2\dot{\beta}(t)(\psi^{3/2} - \psi^{1/2})\cos\theta, \tag{39a}$$

$$\xi_\psi^{(1)}(\psi,\theta,t) = -\frac{R}{4a}\frac{q_0^2}{\psi}\dot{\beta}(t)(3\psi^{3/2} - \psi^{1/2})\sin\theta. \tag{39b}$$

Equations (26) and (39) can be inserted into the time advancement Eq. (3) to give an equation for the evolution of the shift $\Delta(\psi, t)$,

$$\Delta_t = \frac{a^2\dot{\beta}_\theta(t)}{2R}(\psi - 1), \tag{40}$$

where $\dot{\beta}_\theta(t) \equiv q_0^2 R^2 \dot{\beta}(t)/(2a^2)$. Integration of Eq. (40), using Eq. (32c), gives

$$\Delta(\psi, t) = \frac{a^2[1 + 4\beta_\theta(t)]}{8R}(\psi - 1). \tag{41}$$

Finally, we consider now the integration of Eq. (1e). Since $\Psi'_t \sim \Psi'_\psi \sim \varepsilon^2 \Psi'$, we need only keep the two terms

$$\Psi'_t = -\Psi' u_\psi. \tag{42}$$

We now impose the boundary condition that $\Psi'$ (and $g$) match onto the vacuum solution at the plasma boundary. This implies $\Psi'^{(2)} = g^{(2)} = 0$ at $\psi = 1$, which, with Eq. (42), determines the boundary value

$$\Omega_b = \pi a^2 \dot{\beta}_0/(4q_0). \tag{43}$$

Inserting Eqs. (32b), (38), and (43) into Eq. (42) yields

$$\Psi'^{(2)}(\psi, t) = [\beta_0(t)/2][1 - \beta_\theta^{-1}(t)](\psi - 1).$$

In Fig. 1 we present a comparison of the shift of the magnetic axis as computed from the small aspect ratio formula, Eq. (41), and for a full numerical solution with $a/R = 0.1$, 18 $\psi$ zones, 64 $\theta$ zones, and 10 Fourier harmonics. The agreement is excellent for $\beta_\theta \sim 1$, where the ordering assumed in deriving Eq. (41) is valid, and the solutions differ greatly at large values of $\beta_\theta \gg 1$, as is expected since Eq. (41) is no longer valid.

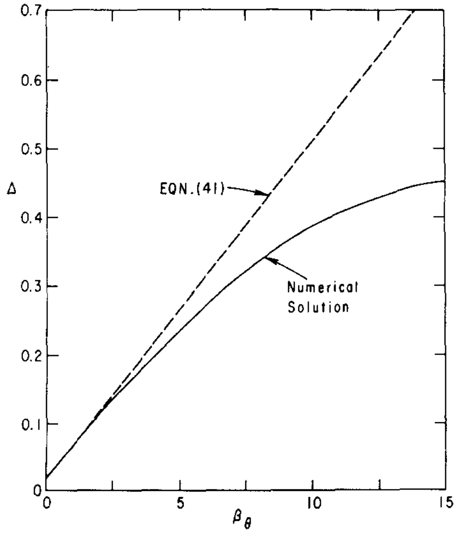


FIG. 1. Comparison of the shift of the magnetic axis predicted by a small aspect ratio expansion, Eq. (41), and by the numerical solution.

## VI. Simulation of a Tokamak Ignition Experiment

The numerical solution procedure described here has been utilized to predict the performance of the proposed ZEPHYR experiment [6]. In the operating scenario for this device, a deuterium–tritium tokamak plasma will be heated to near ignition conditions by intense neutral particle injection, and then will be subjected to major radius compression causing the plasma to ignite.

In addition to the neoclassical transport model described in Appendix C, we have included empirical anomalous electron thermal conduction and particle transport so that $\langle \mathbf{q}_e \cdot \nabla\psi \rangle_{\text{anom}} = -c_e \langle |\nabla\psi|^2 \rangle (T_e)_\psi$ and $\Gamma_{\text{anom}} = -(c_p/n_e)\langle |\nabla\psi|^2 \rangle (n_e)_\psi$ with the constants $c_e$ and $c_p$ equal to $6.25 \times 10^{17}$ and $1.25 \times 10^{17}$ in cgs units. Bremsstrahlung radiation is included as a loss term for the electrons. A diffusion-like neutral gas refilling model [7] has been used to provide a source term in the density equation with the normalization constant chosen to keep the total plasma mass constant during the entire discharge. Previous comparisons with a more involved Monte Carlo

fusion. If we assume the $\alpha$ particles to give up their energy near where they are produced, this can be represented by a local source term [9] (cgs units)

$$S_{p\alpha} = 5.047 \times 10^{-21} n_{\mathrm{D}} n_{\mathrm{T}} \exp\left[-0.476 \left| \ln\left(\frac{T_i}{69}\right)\right|^{2.25}\right]$$

where $n_{\mathrm{D}}$ and $n_{\mathrm{T}}$ are the deuterium and tritium densities (assumed equal) and $T_i$ is the local ion temperature in kiloelectron volts. This source is split 40% to the ions and 60% to the electrons.

Another source term is aded during the time interval when the neutral beam injection is occurring to model the plasma heating due to the slowing down of the fast injection particles, and to the $\alpha$ particles produced by fusion events induced by these high energy injected particles. A source term for the ions in good agreement with that obtained from Monte Carlo calculations [10] is given by

$$S_{pB}(\psi) = P[(\psi/\psi_e)^2 + (d/\psi_e)^2]^{-1}(d/\psi_e)^2[1 - (\psi/\psi_e)^2],$$

where $0 < \psi < \psi_e$ and $(d/\psi_e) = 0.9$. The normalization constant $P$ is chosen so that the total power deposited is $\int_0^{\psi_e} V' S_B(\psi)\, d\psi = 14$ MW during the injection period.

Using these source and transport models, the calculation proceeds as follows. An initial plasma equilibrium is computed with $R = 200$ cm, $a = 60$ cm, toroidal magnetic field at axis $B_T = 61$ kG, and with plasma current $I_p = 2.47$ MA. The calculation proceeds for 0.5 sec with no external heating source, at which time the neutral beam source term is turned on and left on until $t = 1.6$ sec. The major radius compression takes place in the interval 1.5 sec $< t <$ 1.6 sec, and the calculation is halted at $t = 2.0$ sec.

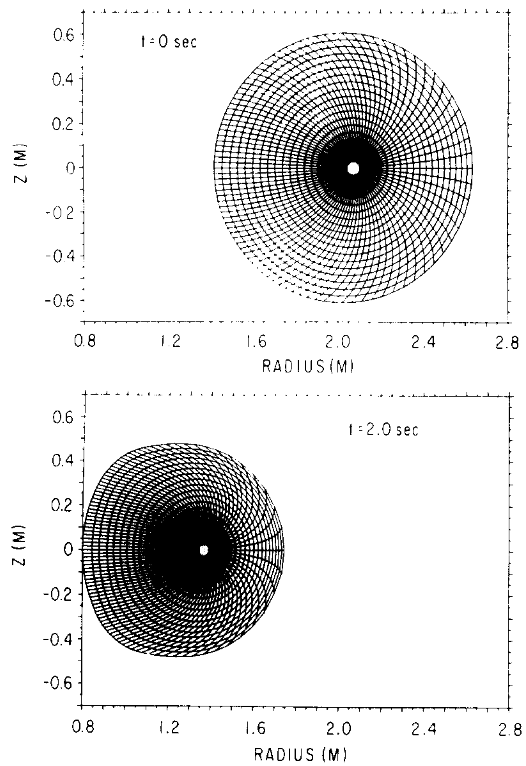


FIG. 2. Constant ($\psi$, $\theta$) coordinate surfaces at (a) $t = 0$ (before compression) and (b) $t = 2$ sec (after compression).

The boundary conditions used in this calculation are as follows. The values of $n$, $T_e$, $T_i$, and $\chi'$ were held at fixed values at the plasma–vacuum boundary (pedestal boundary conditions) so that $n_b = 2 \times 10^{13} (\text{cm})^{-3}$, $T_{ib} = T_{eb} = 30\, eV$, $q_b = 2.68$. The toroidal field in the vacuum is $\mathbf{B}_T = g_v \nabla\phi$ with $g_v$ a constant which gives the value $|B_T| = 61$ kg at $x = 200$ cm. The surface averaged part of the toroidal flux velocity at the boundary, $u_b$, is determined self-consistently so that the equilibrium equation remans satisfied across the plasma–vacuum boundary. The surface varying part of the toroidal flux velocity is prescribed to be zero except during the compression phase at which time it is given a value corresponding to a uniform radially inward velocity.

The results of this calculation are illustrated in Figs. 2 through 5. Figure 2 shows the magnetic surfaces at $t = 0$ (before compression) and at $t = 2$ sec (after compression). During the compression the position of the magnetic axis has decreased from $R = 207$ to 140 cm and the minor radius has decreased from $a = 61$ to 48 cm. Figure 3 shows the midplane values of $T_e$, $T_i$, $n$, and $J_\phi$ at various times during the calculation, and Fig. 4 shows the central values of the temperatures and density as a function of the time $t$.

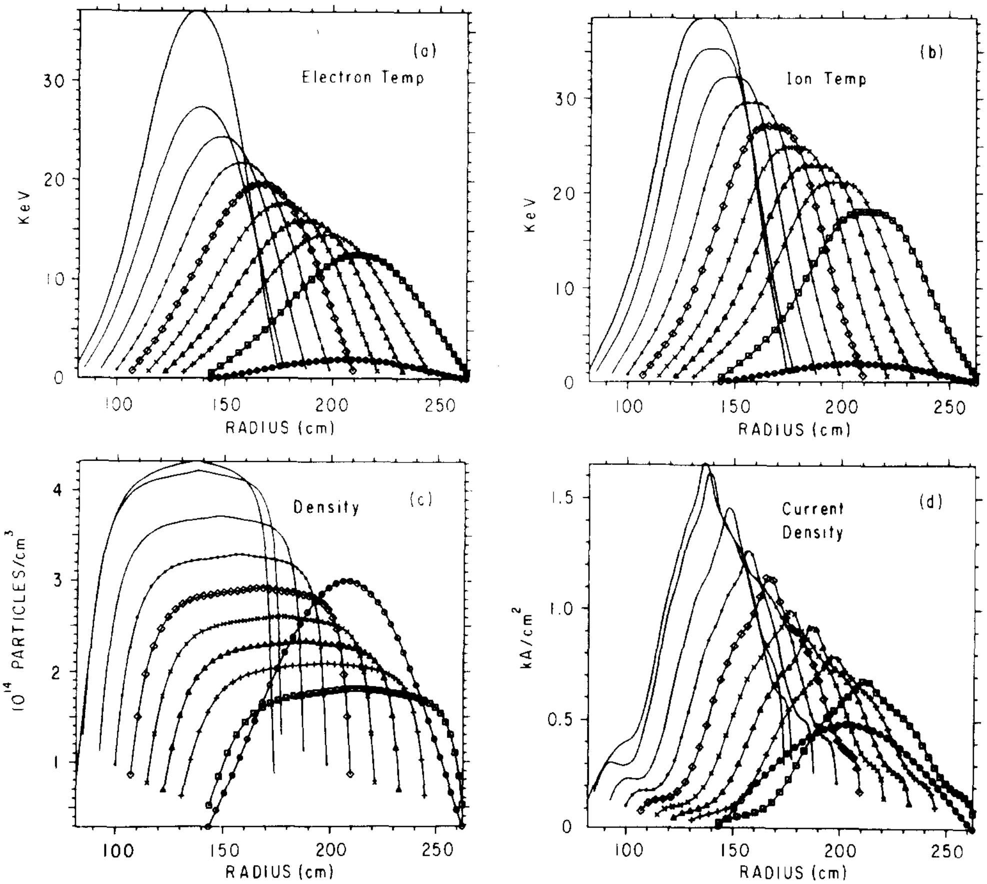


FIG. 3. Profiles of the midplane values of $T_e$, $T_i$, $n$ and $J_\phi$ at time $t = 0.00$, 1.44, 1.51, 1.53, 1.54, 1.56, 1.57, 1.58, 1.60, and 2.00 sec.

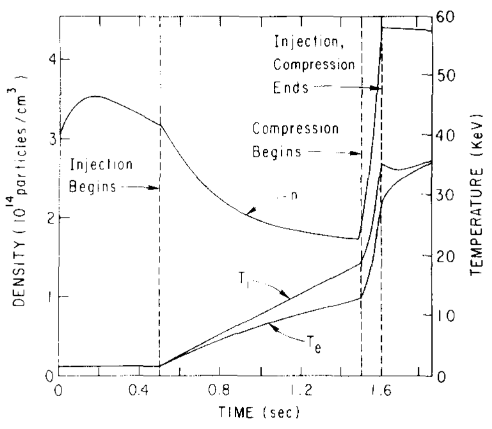


FIG. 4. Time history of the central electron and ion temperatures, $T_e^{(0)}$, $T_i^{(0)}$, and the central density $n^{(0)}$.

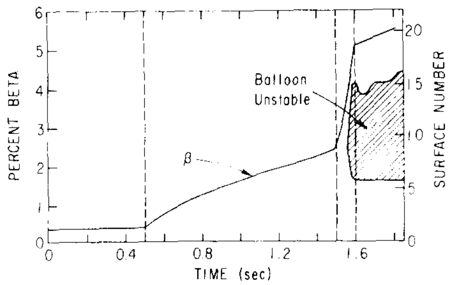


FIG. 5. Time history of the plasma $\beta$ and stability history showing which magnetic surfaces are predicted to be balloon unstable at which times.

Figure 5 presents a graph of the time history of the plasma $\beta$ and the stability history of the plasma to localized ballooning modes. This is obtained by evaluating the ballooning criteria (Appendix D) on each magnetic surface almost continuously during the calculation. The physical consequences of a portion of the plasma being unstable to the linear ballooning criteria is not known at this time, however, it is of interest to note that this experiment should theoretically exceed the ballooning mode limit.

## VII. SUMMARY AND DISCUSSION

A previous paper [1] has presented the complete theory of self-consistent two-dimensional toroidal transport for an electron–ion plasma described by an arbitrary transport model. A reduced set of equations was derived comprised of five one-dimensional evolution equations (1) together with a single two-dimensional boundary value equation (2). Magnetic flux coordinates are used to describe the diffusion of plasma quantities through magnetic surfaces of changing shape.

This paper has described in detail a general method for the numerical solution of the equations derived in Ref. [1]. The relative immobility of the constant toroidal flux surfaces is exploited by the present formulation in that the one-dimensional system of evolution equation is only weakly coupled to the two-dimensional equation determining the velocity of the toroidal flux surfaces. This allows the separate numerical solution of the one-dimensional and of the two-dimensional equations each time step. The one-dimensional vector system of surface averaged transport equations is solved implicitly resulting in the inversion of a block tridiagonal matrix each time step with size $4 \times 4$ blocks. This matrix inversion method allows treatment of an arbitrary transport model with any number of mixed derivatives.

The two-dimensional equation to be solved each time step for the toroidal flux velocity is linear and can thus be solved by Fourier analyzing in the angle coordinate. After finite differencing the transformed two-dimensional equation, a block tridiagonal system is again obtained, identical in structure to that of the one-dimensional evolution equations. The size of the blocks here is determined by the number of Fourier modes retained in the expansion. For moderately noncircular plasma cross sections, a small number of modes (five or fewer) is usually adequate. Since the two-dimensional equation to be solved reduces to a form mathematically equivalent to the system of one-dimensional equations, it is evident that the amount of computer time required to solve a two-dimensional problem is only a small multiple of that required to solve a one-dimensional problem.

The solution method presented here is flexible in that it yields fully implicit solutions of arbitrary transport models with the off diagonal terms in the transport matrix being treated implicitly on an equal footing with the diagonal terms. A full neoclassical model thus fits into the solution framework as do simple empirical models. Arbitrary additional energy and particle source terms may be inserted into the conservation equations to model such effects as neutral gas influx, charge exchange, radiation, and neutral beam heating, without affecting the solution procedure.

The formulation and solution procedure outlined here is fundamentally different from other [11–16] published solution algorithms for the two-dimensional tokamak transport problem. These procedures alternate solving the one-dimensional surface averaged transport equations with a step in which the solution is mapped onto a rectangular grid on which the nonlinear equilibrium (Grad–Shafranov) equation is solved. While these methods may seem conceptually more straightforward, the present method avoids the truncation error inherent in mapping onto an Eulerian grid and eliminates the need for repeated solutions of a nonlinear elliptic equation.

## Appendix A

The matrix elements of $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{C}_j$, and $\mathbf{D}_j$ used in advancing the one-dimensional surface averaged transport equations, Eq. (16), are as follows:

$$a_{11} = s_1(u_{j+1/2} - \beta^0_{j+1/2}) - s_2(\beta^1 N')_{j+1/2}/V'_{j+1},$$

$$c_{11} = -s_1(u_{j-1/2} - \beta^0_{j-1/2}) - s_2(\beta^1 N')_{j-1/2}/V'_{j-1},$$

$$b_{11} = -1 - s_1(u_{j+1/2} - u_{j-1/2} - \beta^0_{j+1/2} + \beta^0_{j-1/2}) - s_2[(\beta^1 N')_{j+1/2} + (\beta^1 N')_{j-1/2}]/V'_j,$$

$$a_{12} = -s_2(N'\beta^2)_{j+1/2}/V'_{j+1},$$

$$c_{12} = -s_2(N'\beta^2)_{j-1/2}/V'_{j-1},$$

$$b_{12} = -s_2[(N'\beta^2)_{j+1/2} + (N'\beta^2)_{j-1/2}]/V'_j,$$

$$a_{13} = -s_2(N'\beta^3)_{j+1/2}/V'_{j+1},$$

$$c_{13} = -s_2(N'\beta^3)_{j-1/2}/V'_{j-1},$$

$$a_{14} = -s_2(N'\beta^4)_{j+1/2}\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1},$$

$$c_{14} = -s_2(N'\beta^4)_{j-1/2}\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1},$$

$$b_{14} = -s_2[(N'\beta^4)_{j+1/2} + (N'\beta^4)_{j-1/2}]\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_j,$$

$$d_1 = -N_j'^n + \Delta t(1-\theta)[N'(u-\beta)]^n_{\psi j} - \Delta t\, V'\langle S_n\rangle,$$

$$a_{21} = s_2 s_3[(\gamma^1 + \delta^1)]_{j+1/2}/V'_{j+1},$$

$$c_{21} = s_2 s_3[(\gamma^1 + \delta^1)]_{j-1/2}/V'_{j-1},$$

$$b_{21} = s_2 s_3\{[(\gamma^1 + \delta^1)]_{j+1/2} + [(\gamma^1 + \delta^1)]_{j-1/2}\}/V'_j,$$

$$a_{22} = s_1 u_{j+1/2} + s_2 s_3[(\gamma^2 + \delta^2)]_{j+1/2}/V'_{j+1},$$

$$c_{22} = -s_1 u_{j-1/2} + s_2 s_3[(\gamma^2 + \delta^2)]_{j-1/2}/V'_{j-1},$$

$$b_{22} = -1 - s_1(u_{j+1/2} - u_{j-1/2}) + s_2 s_3\{[(\gamma^2 + \delta^2)]_{j+1/2} + [(\gamma^2 + \delta^2)]_{j-1/2}\}/V'_j,$$

$$a_{23} = s_2 s_3[(\gamma^3 + \delta^3)]_{j+1/2}/V'_{j+1},$$

$$c_{23} = s_2 s_3[(\gamma^3 + \delta^3)]_{j-1/2}/V'_{j-1},$$

$$b_{23} = s_2 s_3\{[(\gamma^3 + \delta^3)]_{j+1/2} + [(\gamma^3 + \delta^3)]_{j-1/2}\}/V'_j,$$

$$a_{24} = s_2 s_3[(\gamma^4 + \delta^4)]_{j+1/2}\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1} - \frac{1}{16\pi^3}s_1 s_3 \alpha_j\left(V'\left\langle\frac{\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1/2},$$

$$c_{24} = s_2 s_3[(\gamma^4 + \delta^4)]_{j-1/2}\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1} + \frac{1}{16\pi^3}s_1 s_3 \alpha_j\left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1/2},$$

$$b_{24} = s_2 s_3\{[(\gamma^4 + \delta^4)]_{j+1/2} + [(\gamma^4 + \delta^4)]_{j-1/2}\}\left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_j$$
$$- \frac{s_1 s_3 \alpha_j}{16\pi^3}\left[\left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1/2} - \left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)\right]_{j-1/2},$$

$$d_2 = -\sigma_j'^n + 2s_1 s_3\{[(\gamma^0 + \delta^0)]_{j+1/2} - [(\gamma^0 + \delta^0)]_{j-1/2}\}$$
$$+ \Delta t(1-\theta)\left\{(\sigma' u)_\psi + s_3[(\gamma + \delta)]_\psi - \frac{s_3 \alpha}{16\pi^3}\left(\chi' V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_\psi\right\}^n_j$$
$$- \Delta t\, s_3 V'\langle S_p\rangle,$$

$$a_{31} = s_2 s_4(\delta^1)_{j+1/2}/V'_{j+1} + s_1 s_4(\varepsilon^1)_j/V'_{j+1},$$

$$c_{31} = s_2 s_4(\delta^1)_{j-1/2}/V'_{j-1} - s_1 s_4(\varepsilon^1)_j/V'_{j-1},$$

$$b_{31} = s_2 s_4[(\delta^1)_{j+1/2} + (\delta^1{}_{j-1/2})/V'_j,$$

$$a_{32} = s_2 s_4(\delta^2)_{j+1/2}/V'_{j+1} - 2/3 s_1 \beta V'(p_e/p)^{-2/5}/V'_{j+1} + s_1 s_4(\varepsilon^2)_j/V'_{j+1},$$

$$c_{32} = s_2 s_4 (\delta^2)_{j-1/2}/V'_{j-1} + 2/3 s_1 \beta V' (p_e/p)^{-2/5}/V'_{j-1} - s_1 s_4 (\varepsilon^2)_j/V'_{j-1},$$

$$b_{32} = s_2 s_4 [(\delta^2_{j+1/2} + \delta^2_{j-1/2}]/V'_j,$$

$$a_{33} = s_1 u_{j+1/2} + s_2 s_4 (\delta^3)_{j+1/2}/V'_{j+1} + 2/3 s_1 (\beta V')_j/V'_{j+1} + s_1 s_4 (\varepsilon^3)_j/V'_{j+1},$$

$$c_{33} = -s_1 u_{j-1/2} + s_2 s_4 (\delta^3)_{j-1/2}/V'_{j-1} - 2/3 s_1 (\beta V')_j/V'_{j-1} - s_1 s_4 (\varepsilon^3)_j/V'_{j-1},$$

$$b_{33} = -1 - s_1 (u_{j+1/2} - u_{j-1/2}) + s_2 s_4 \{(\delta^3)_{j+1/2} + (\delta^3)_{j-1/2}\}/V'_j - 2/5 H_j \Delta t [2(1-\phi) + (p/p_e)(2/3+\phi)],$$

$$a_{34} = s_2 s_4 (\delta^4)_{j+1/2} \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1} - \frac{1}{16\pi^3} s_1 s_4 \alpha_j \left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1/2} + s_1 s_4 (\varepsilon^4)_j \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1/2},$$

$$c_{34} = s_2 s_4 (\delta^4)_{j-1/2} \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1} + \frac{1}{16\pi^3} s_1 s_4 \alpha_j \left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1/2} - s_1 s_4 (\varepsilon^4)_j \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1/2},$$

$$b_{34} = s_2 s_4 [(\delta^4)_{j+1/2} + (\delta^4)_{j-1/2}]] \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_j + \frac{s_1 s_4 \alpha_j}{16\pi^3}\left[\left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1/2} - \left(V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1/2}\right],$$

$$d_3 = -\sigma'^n_{ej} + 2 s_1 s_4 [(\delta^0)_{j+1/2} - (\delta^0)_{j-1/2}] + \Delta t\, \theta s_4 \varepsilon^0_j + \Delta t(1-\theta)\left[(\sigma'_e u)_\psi + s_4(\delta_\psi + \varepsilon) - \frac{s_4 \alpha}{16\pi^3}\left(\chi' V'\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_\psi - s_4 \beta V'(p_i)_\psi\right]^n_j - \Delta t\, s_4 V' \langle S_{pe}\rangle + 2/5 H_j \Delta t [2\phi - (p/p_e)(5/3+\phi)]_j,$$

$$a_{41} = -s_2 \alpha^1_{j+1/2}/V'_{j+1},$$

$$c_{41} = -s_2 \alpha^1_{j-1/2}/V'_{j-1},$$

$$b_{41} = -s_2(\alpha^1_{j+1/2} + \alpha^1_{j-1/2})/V'_j,$$

$$a_{42} = -s_2 \alpha^2_{j+1/2}/V'_{j+1},$$

$$c_{42} = -s_2 \alpha^2_{j-1/2}/V'_{j-1},$$

$$b_{42} = -s_2(\alpha^2_{j+1/2} + \alpha^2_{j-1/2})/V'_j,$$

$$a_{43} = -s_2 \alpha^3_{j+1/2}/V'_{j+1},$$

$$c_{43} = -s_2 \alpha^3_{j-1/2}/V'_{j-1},$$

$$b_{43} = -s_2(\alpha^3_{j+1/2} + \alpha^3_{j-1/2})/V'_j,$$

$$a_{44} = s_1 u_{j+1/2} - s_2 \alpha^4_{j+1/2} \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1},$$

$$c_{44} = -s_1 u_{j+1/2} - s_2 \alpha^4_{j-1/2} \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j-1},$$

$$b_{44} = -1 - s_1(u_{j+1/2} - u_{j-1/2}) - s_2 \left(\alpha^4_{j+1/2} + \alpha^4_{j-1/2}\right) \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_j,$$

$$d_4 = -\chi_j'^n - 2s_1(\alpha^0_{j+1/2} - \alpha^0_{j-1/2}) + \Delta t(1-\theta)[(\chi' u - \alpha)_\psi]^n_j.$$

Here

$$s_1 = \theta\, \Delta t/(2\, \Delta\psi),$$
$$s_2 = \theta\, \Delta t/(\Delta\psi)^2,$$
$$s_3 = 2/5(\sigma'/pV')^n_j,$$
$$s_4 = 2/5(\sigma'_e/p_e V')^n_j,$$
$$H_j = 3\eta_{\perp j}(4\pi n_j e^2/M_i c^2),$$
$$\phi = 5/2.$$

Most of the above expressions for the matrix elements are valid at the first surface where

$$u_{j-1/2} = \alpha^0_{j-1/2} \cdots \alpha^3_{j-1/2} = \beta^0_{j-1/2} \cdots \beta^4_{j-1/2} = \alpha^0_{j-1/2} \cdots \alpha^4_{j-1/2}$$
$$= \delta^0_{j-1/2} \cdots \delta^4_{j-1/2} = \varepsilon^0_{j-1/2} \cdots \varepsilon^4_{j-1/2} = 0.$$

Special care must be taken in the evaluation of $a_{44}$, $c_{44}$, and $b_{44}$, at the origin. Taylor expanding gives the result

$$\left[\frac{1}{V'}\left(\frac{V'\chi'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_\psi\right]_{\psi=0} = \frac{4\pi}{gx^2}\left(\frac{2}{3\,\Delta\psi}\right)^l [3^{2l+1}\chi'_{j=1} - \chi'_{j=2}]/(3^{l+1}-1).$$

Thus, at $j = 1$

$$a_{44} = s_1 u_{j+1/2} - s_2 \alpha^4_{j-1/2} \left(\frac{V'}{g}\left\langle\frac{|\nabla\psi|^2}{x^2}\right\rangle\right)_{j+1} + s_5(\alpha^4 V')_{j-1/2},$$

$$c_{44} = 0,$$

$$b_{44} = -s_1 u_{j+1/2} - s_2 \alpha^4_{j+1/2} + 3^{2l+1} s_5 (\alpha^4 V')_{j-1/2},$$

$$s_5 = \frac{4\pi}{\Delta\psi\, gx^2}\left(\frac{2}{3\,\Delta\psi}\right)^l \frac{1}{(3^{l+1}-1)}.$$

## APPENDIX B

The elements of $\mathbf{A}_j$, $\mathbf{B}_j$, and $\mathbf{C}_j$ needed for the two-dimensional equation (24) describing the evolution of the toroidal flux surfaces, are defined as follows:

$$a_j^{0,0} = \tilde{a}_j^{0,0} + \frac{1}{2\,\Delta\psi^2}\, b_{j+1/2}^0,$$

$$c_j^{0,0} = \tilde{c}_j^{0,0} + \frac{1}{2\,\Delta\psi^2}\, b_{j-1/2}^0,$$

$$b_j^{0,0} = \tilde{b}_j^{0,0} + \frac{1}{2\,\Delta\psi^2}\,(b_{j+1/2}^0 + b_{j-1/2}^0) - 1/2 d_j^0.$$

For $q > 0$,

$$a_j^{q,0} = \tilde{a}_j^{q,0} + \frac{1}{\Delta\psi^2}\, b_{j+1/2}^q + \frac{1}{2\,\Delta\psi}\, q a_j^q,$$

$$c_j^{q,0} = \tilde{c}_j^{q,0} + \frac{1}{\Delta\psi^2}\, b_{j-1/2}^q - \frac{1}{2\,\Delta\psi}\, q a_j^q,$$

$$b_j^{q,0} = \tilde{b}^{q,0} + \frac{1}{\Delta\psi^2}\,(b_{j+1/2}^q + b_{j-1/2}^q) - d_j^q.$$

For $m = 1, q$ and $q \geqslant 0$,

$$a_j^{q,m} = \tilde{a}_j^{q,m} + \tilde{\tilde{a}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\, b_{j+1/2}^{q-m} + \frac{1}{4\,\Delta\psi}\,(m a_{j+1}^{-m+q} + q a_j^{q-m}),$$

$$c_j^{q,m} = \tilde{c}_j^{q,m} + \tilde{\tilde{c}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\, b_{j-1/2}^{q-m} - \frac{1}{4\,\Delta\psi}\,(m a_{j-1}^{-m+q} + q a_j^{q-m}),$$

$$b_j^{q,m} = \tilde{b}_j^{q,m} + \tilde{\tilde{b}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\,(b_{j+1/2}^{q-m} + b_{j-1/2}^{q-m}) + \frac{1}{2}\, q m c_j^{q-m} - \frac{1}{2}\, d_j^{q-m}.$$

For $m = q + 1, M$ and $q \geqslant 0$,

$$a_j^{q,m} = \tilde{a}_j^{q,m} + \tilde{\tilde{a}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\, b_{j+1/2}^{m-q} - \frac{1}{4\,\Delta\psi}\,(m a_{j+1}^{m-q} + q a_j^{m-q}),$$

$$c_j^{q,m} = \tilde{c}_j^{q,m} + \tilde{\tilde{c}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\, b_{j-1/2}^{m-q} + \frac{1}{4\,\Delta\psi}\,(m a_{j-1}^{m-q} + q a_j^{m-q}),$$

$$b_j^{q,m} = \tilde{b}_j^{q,m} + \tilde{\tilde{b}}_j^{q,m} + \frac{1}{2\,\Delta\psi^2}\,(b_{j+1/2}^{m-q} + b_{j-1/2}^{m-q}) + \frac{1}{2}\, q m c_j^{m-q} - \frac{1}{2}\, d_j^{m-q}.$$

Here

$$\tilde{a}_j^{q,m} = (2-\delta_{q0})\frac{1}{4\Delta\psi^2}(H_j^q P_{j+1/2} H_{j+1}^m + J_j^q G_{j+1/2} J_{j+1}^m),$$

$$\tilde{c}_j^{q,m} = (2-\delta_{q0})\frac{1}{4\,\Delta\psi^2}(H_j^q P_{j-1/2} H_{j-1}^m + J_j^q G_{j-1/2} J_{j-1}^m),$$

$$\tilde{b}_j^{q,m} = (2-\delta_{q0})\frac{1}{4\,\Delta\psi^2}(H_j^q(P_{j+1/2}+P_{j-1/2})\,H_j^m + J_j^q(G_{j+1/2}+G_{j-1/2})\,J_j^m),$$

where $\delta_{jk}$ is the Kronecker delta. For $1 \leqslant m \leqslant M-q$,

$$\tilde{\tilde{a}}_j^{q,m} = \frac{1}{4\,\Delta\psi}(-ma_{j+1}^{m+q} + qa_j^{m+q}) + \frac{1}{2\,\Delta\psi^2} b_{j+1/2}^{m+q},$$

$$\tilde{\tilde{c}}_j^{q,m} = \frac{1}{4\,\Delta\psi}(ma_{j-1}^{m+q} - qa_j^{m+q}) + \frac{1}{2\,\Delta\psi^2} b_{j-1/2}^{m+q},$$

$$\tilde{\tilde{b}}_j^{q,m} = -\frac{1}{2}qmc_j^{m+q} - \frac{1}{2}d_j^{m+q} + \frac{1}{2\,\Delta\psi^2}(b_{j+1/2}^{m+q} + b_{j-1/2}^{m+q}).$$

For $m+q > M$ we have $\tilde{\tilde{a}}_j^{q,m} = \tilde{\tilde{c}}_j^{q,m} = \tilde{\tilde{b}}_j^{q,m} = 0$.
Special care is taken at $\psi = 0$ ($j = 0$), where we note

$$J_0^0 = \left(\frac{3\,\Delta\psi}{2}\right)^l \frac{1}{2\pi x_0}(3^{l+1}-1)/(3^{2l+1}\chi_1' - \chi_2'),$$

$$H_0^0 = x_0^2 J_0^0,$$

$$b_0^0 = 0.$$

Also, of course, $a_0^m = b_0^m = c_0^m = J_0^m = H_0^m = 0$ for $m > 0$.

## Appendix C

In Section VIII of Ref. [1] expressions were given for the resistive transport fluxes for an electron/ion plasma for the collision dominated regime (Pfirsch–Schluter regime) and for the long mean free path banana regime. Here we decribe how these are incorporated into the present formalism by giving explicit expressions for the $\alpha$, $\beta$, $\gamma$, $\delta$, and $\varepsilon$ defined in Section III.

### A. *Pfirsch–Schlüter Regime*

$$\alpha^4 = \eta_{\parallel}\, g/(4\pi V'\langle x^{-2}\rangle),$$

$$\beta^1 = -\frac{p_e}{n} L_{12},$$

$$\beta^2 = -\frac{5}{3}\frac{V'p}{\sigma'} L_{11},$$

$$\beta^3 = \frac{5}{3}\frac{V'p_e}{\sigma'} L_{12},$$

$$\beta^4 = -\alpha^4 \frac{\langle B_p^2 \rangle}{\langle B^2 \rangle \chi'},$$

$$\gamma^1 = V'p_i \left( L_i \frac{p_i}{n} + \frac{5}{2}\beta^1 \right),$$

$$\gamma^2 = V'p_i \left( -\frac{5}{3}\frac{V'p}{\sigma'} L_i + \frac{5}{2}\beta^2 \right),$$

$$\gamma^3 = V'p_i \left( \frac{5}{3}\frac{V'p_e}{\sigma'_e} L_i + \frac{5}{2}\beta^3 \right),$$

$$\gamma^4 = V'p_i \frac{5}{2}\beta^4,$$

$$\delta^1 = V'p_e \left( L_{22} \frac{p_e}{n} + \frac{5}{2}\beta^1 \right),$$

$$\delta^2 = V'p_e \left( L_{12} \frac{5}{3}\frac{V'p}{\sigma'} + \frac{5}{2}\beta^2 \right),$$

$$\delta^3 = V'p_e \left( -L_{22} \frac{5}{3}\frac{V'p_e}{\sigma'_e} + \frac{5}{3}\beta^3 \right),$$

$$\delta^4 = V'p_e \frac{5}{2}\beta^4,$$

$$\alpha^0 = \alpha^1 = \alpha^2 = \alpha^3 = \beta^0 = \gamma^0 = \delta^0 = \varepsilon^0 = \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = \varepsilon^4 = 0.$$

Here the transport coefficients are

$$L_{11} = L_0[1 + 2.65(\eta_{\parallel}/\eta_{\perp})\, q_*^2],$$

$$L_{12} = 3/2L_0[1 + 1.47(\eta_{\parallel}/\eta_{\perp})\, q_*^2],$$

$$L_{22} = 4.66L_0[1 + 1.67(\eta_{\parallel}/\eta_{\perp})\, q_*^2],$$

$$L_i = \sqrt{2}\, L_0(m_i/m_e)^{1/2}(T_e/T_i)^{3/2}(1 + 1.60q_*^2),$$

$$L_0 = \eta_{\perp} \left\langle \frac{|\nabla\psi|^2}{B^2} \right\rangle,$$

$$\eta_\perp = m_e(ne^2\tau_{ei})^{-1},$$

$$q_*^2 = \frac{1}{2} g^2 \left\langle \frac{|\nabla\psi|^2}{B^2} \right\rangle^{-1} (\langle B^{-2}\rangle - \langle B^2\rangle^{-1}),$$

$$\eta_\| = 0.51\eta_\perp.$$

### B. *Banana Regime*

$$\alpha^1 = \frac{2\pi L_{33}^e}{g\langle x^{-2}\rangle} (L_{23}^e T_e - yL_{13}^e T_i),$$

$$\alpha^2 = \left(\frac{5}{3}\right) \frac{2\pi V'p}{g\langle x^{-2}\rangle \sigma'} L_{33}^e L_{13}^e (1+y),$$

$$\alpha^3 = -\left(\frac{5}{3}\right) \frac{2\pi V'p_e}{g\langle x^{-2}\rangle \sigma_e'} L_{33}^e (L_{23}^e + yL_{13}^e),$$

$$\alpha^4 = \frac{L_{33}^e\, g}{4\pi\langle x^{-2}\rangle V'},$$

$$\beta^1 = (L_{33}^{e*} L_{13}^e + L_{11}^{nc_e})\, yT_i - (L_{33}^{e*} L_{23}^e + L_{12}^e)\, T_e,$$

$$\beta^2 = -\frac{5}{3}\frac{V'p}{\sigma'} [L_{33}^{e*} L_{13}^e (1+y) + L_{11}^{nc_e} y + L_{11}^e],$$

$$\beta^3 = \frac{5}{3}\frac{V'p_e}{\sigma_e'} [L_{33}^{e*}(L_{13}^e\, y + L_{23}^e) + L_{11}^{nc_e} y + L_{12}^e],$$

$$\beta^4 = -\frac{L_{33}^{e*}\, g^2}{2\pi V'},$$

$$\gamma^1 = V'p_i \left(L_{22}^i T_i + \frac{5}{2}\beta^1\right),$$

$$\gamma^2 = V'p_i \left(-\frac{5}{3}\frac{V'p}{\sigma'} L_{22}^i + \frac{5}{2}\beta^2\right),$$

$$\gamma^3 = V'p_i \left(\frac{5}{3}\frac{V'p_e}{\sigma_e'} L_{22}^i + \frac{5}{2}\beta^3\right),$$

$$\gamma^4 = V'p_i 5/2\beta^4,$$

$$\delta^1 = V'p_e \left[\frac{L_{23}^e L_{33}^e}{\langle B^2\rangle} (L_{23}^e T_e - L_{13}^e T_i y) + L_{22}^e T_e - yL_{12}^{nc_e} T_i + \frac{5}{2}\beta^1\right],$$

$$\delta^2 = V'p_e \left\{\frac{5}{3}\frac{V'p}{\sigma'} \left[\frac{L_{23}^e L_{33}^e L_{13}^e}{\langle B^2\rangle} (1+y) + L_{12}^e + yL_{12}^{nc_e}\right] + \frac{5}{2}\beta^2\right\},$$

$$\delta^3 = V'p_e \left\{ -\frac{5}{3}\frac{V'p_e}{\sigma'_e}\left[\frac{L^e_{23}L^e_{33}}{\langle B^2\rangle}(L^e_{13}\,y + L^e_{23}) + L^{nc_e}_{12}y + L^e_{22}\right] + \frac{5}{2}\beta^3\right\},$$

$$\delta^4 = V'p_e\left(\frac{L^e_{23}L^e_{33}\,g^2}{2\pi V'\langle B^2\rangle} + \frac{5}{2}\beta^4\right),$$

$$\varepsilon^1 = -y\beta\,\frac{p_i V'}{n},$$

$$\varepsilon^2 = y\beta\,\frac{5}{3}\frac{(V')^2 p}{\sigma'},$$

$$\varepsilon^3 = -y\beta\,\frac{5}{3}\frac{(V')^2 p_e}{\sigma'_e},$$

$$\alpha^0 = \beta^0 = \gamma^0 = \delta^0 = \varepsilon^0 = \varepsilon^4 = 0.$$

Here

$$L^i_{22} = \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{1/2}[1.41L_0(1+2q_*^2) + 0.65L_*(1-0.5f_t)^{-1}],$$

$$L^{nc_e}_{11} = L_*(1.53 - 0.53f_t),$$

$$L^{nc_e}_{12} = L_*(2.13 - 0.63f_t),$$

$$L^e_{11} = L_0(1+2q_*^2) + L^{nc_e}_{11},$$

$$L^e_{12} = 3/2L_0(1+2q_*^2) + L^{nc_e}_{12},$$

$$L^e_{22} = 4.66[L_0(1+2q_*^2) + L_*],$$

$$L^e_{23} = 1.27\left(\frac{2\pi g}{\chi'}\right)f_t(1-f_t),$$

$$L^e_{13} = 1.66\left(\frac{2\pi g}{\chi'}\right)f_t(1-0.38f_t),$$

$$L^e_{33} = 0.51\eta_\perp[1 - 1.26f_t(1-0.18f_t)]^{-1},$$

$$L_* = \eta_\perp\left(\frac{2\pi g}{\chi'}\right)^2 f_t\langle B^2\rangle^{-1},$$

$$L_0 = \eta_\perp\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle,$$

$$q_*^2 = \frac{1}{2}\left(\frac{2\pi g}{\chi'}\right)^2(\langle B^{-2}\rangle - \langle B^2\rangle^{-1})\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle,$$

$$y = -1.17(1-f_t)(1-0.54f_t)^{-1},$$

$$\eta_\perp = \frac{m_e}{ne^2\tau_{ei}},$$

$$f_t = 1 - \frac{3}{4}\langle B^2\rangle \int_0^{B_c^{-1}} \frac{\lambda\, d\lambda}{\sqrt{1-\lambda B\rangle}},$$

$B_c$ = maximum value of $|B|$ on a flux surface,

$$L_{33}^{e*} = \left(g\frac{\langle B_p^2\rangle}{\langle B_T^2\rangle} + L_{13}^e\right)\frac{2\pi L_{33}^e}{\chi'\langle B^2\rangle}.$$

### C. *Anamalous Transport*

The standard empirical anomalous transport leads to local particle and electron energy confinement times scaling like $na^2/c_p$ and $na^2/c_e$, where $a$ is the minor radius and $c_p$ and $c_e$ are constants. This can be included in the present formalism by adding the following terms to the neoclassical model:

$$\beta^1 = \beta_{nc}^1 - c_p\langle|\nabla\psi|^2\rangle/n^2,$$

$$\delta^1 = \delta_{nc}^1 - \left(\frac{5}{2}c_p - c_e\right)V'\langle|\nabla\psi|^2\rangle p_e/n^2,$$

$$\delta^3 = \delta_{nc}^3 - \frac{5}{3}c_e V'^2\langle|\nabla\psi|^2\rangle p_e/\sigma n.$$

## Appendix D

The linear stability of a plasma equilibrium to ballooning and to interchange modes can be determined by applying the local stability criterion [17] to individual magnetic surfaces. For ballooning modes, this involves integrating an ordinary differential equation in the poloidal angle. For interchange modes, this criterion involves only the evaluation of an expression involving surface averages of equilibrium quantities.

The Euler equation for $k = 0$ ballooning instabilities reads

$$\frac{\partial}{\partial\theta_H}\left(\frac{\chi'^2}{B^2}|\nabla\beta|^2\frac{\partial\Phi}{\partial\theta_H}\right) + \frac{p'}{\chi'}2\kappa_w\Phi = 0.$$

$$2\kappa_w = -\nabla\cdot\left(\frac{\mathbf{B}\times\nabla\beta}{B^2}\right) + \frac{\mathbf{B}\cdot\nabla\beta\times\nabla p}{B^4}.$$

In an axisymmetric coordinate system with $J = x^m \psi^l$ this gives

$$\frac{\partial}{\partial \theta_H} = \frac{1}{2\pi J} \frac{\partial}{\partial \theta},$$

$$|\nabla \beta|^2 = \frac{1}{(2\pi)^2} \left( \frac{1}{x^2} + q^2 |\nabla \theta|^2 + 2qq'\theta \nabla \theta \cdot \nabla \psi + q'\theta^2 |\nabla \psi|^2 \right),$$

$$2\kappa_w = \frac{-1}{J} \left[ \frac{\partial}{\partial \psi} \left( \frac{J}{\chi'} \right) + \frac{\partial}{\partial \theta} \frac{1}{B^2} \left( \frac{J\chi'}{(2\pi)^2 x^2} \nabla \theta \cdot \nabla \psi - \frac{gq'\theta}{2\pi} \right) \right] + \frac{p'}{\chi' B^2}.$$

We can define the three coefficients

$$\alpha_1 = \frac{\chi'^4}{(2\pi)^3 J B^2} \left( \frac{1}{x^2} + q^2 |\nabla \theta|^2 + 2qq'\theta \nabla \theta \cdot \nabla \psi + q'^2 \theta^2 |\nabla \psi|^2 \right),$$

$$\alpha_2 = \frac{p'}{B^2} \left[ \frac{J\chi'^2}{(2\pi)\, x^2} \nabla \theta \cdot \nabla \psi - \chi' g q' \theta \right],$$

$$\alpha_3 = 2\pi p' \left[ \frac{Jp'}{B^2} - \chi' \frac{\partial}{\partial \psi} (J/\chi') \right].$$

The Euler equation then becomes

$$\alpha_1 \frac{\partial y_1}{\partial \theta} = \alpha_2 y_1 + y_2,$$

$$\alpha_1 \frac{\partial y_2}{\partial \theta} = -(\alpha_3 \alpha_1 + \alpha_2^2)\, y_1 - \alpha_2 y_2.$$

For an up–down symmetric problem, we set $y_1(0) = 1$, $y_2(0)$ and integrate outward from the origin. If the function $y_1(\theta)$ ever changes sign (becomes negative), this corresponds to an instability.

The stability of a plasma with respect to resistive and ideal interchange modes can be determined by evaluating the three surface functions [18]:

$$E = \frac{\langle B^2/|\nabla Z|^2 \rangle V'}{\Lambda^2} \left( K' \Psi'' - I' \chi'' - p' V'' + \frac{\Lambda \langle \sigma B^2 \rangle}{\langle B^2 \rangle} \right),$$

$$F = \frac{\langle B^2/|\nabla Z|^2 \rangle V'^2}{\Lambda^2} \left( \left\langle \frac{\sigma^2 B^2}{|\nabla Z|^2} \right\rangle - \frac{\langle \sigma B^2/|\nabla Z|^2 \rangle^2}{\langle B^2/|\nabla Z|^2 \rangle} + p'^2 \left\langle \frac{1}{B^2} \right\rangle \right),$$

$$H = \frac{V'}{\Lambda \langle B^2 \rangle} \left( \langle B^2 \rangle \left\langle \frac{\sigma B^2}{|\nabla Z|^2} \right\rangle - \langle \sigma B^2 \rangle \left\langle \frac{B^2}{|\nabla Z|^2} \right\rangle \right).$$

Here, $\Lambda \equiv \Psi' \chi'' - \chi' \Psi''$, $\Psi(Z)$ and $\chi(Z)$ are the toroidal and the poloidal magnetic fluxes, $I(Z)$ and $K(Z)$ are the poloidal and toroidal current fluxes, $\sigma \equiv \mathbf{J} \cdot \mathbf{B}/B^2$, $Z$ is an arbitrary surface label, and primes denote differentiation with respect to $Z$.

On each magnetic surface, the quantities

$$D_I = E + F + H - (1/4),$$
$$D_R = E + F + H^2,$$

are evaluated. The plasma is unstable to ideal or resistive interchanges if $D_I > 0$ or $D_R > 0$. Choosing $Z = \psi$ and specializing to an axisymmetric system, the functions $E$, $F$, $H$ are evaluated as

$$E = \frac{V'p'}{\chi'^3 q'^2} \left\langle \frac{B^2}{|\nabla\psi|^2} \right\rangle \left( \frac{2\pi q' g}{\langle B^2 \rangle} - \left( \frac{V'}{\chi'} \right)' \right),$$

$$F = \frac{V'^2 p'^2}{q'^2 \chi'^4} \left\{ - \left( \frac{2\pi g}{\chi'} \right)^2 \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle^2 + \left\langle \frac{x^2}{|\nabla\psi|^2} \right\rangle \right.$$
$$\left. \times \left[ \left( \frac{2\pi g}{\chi'} \right)^2 \left\langle \frac{1}{x^2 |\nabla\psi|^2} \right\rangle + \left\langle \frac{1}{x^2} \right\rangle \right] \right\},$$

$$H = \frac{-V' g p'}{(2\pi) \chi' q' \langle B^2 \rangle} \left\{ \left\langle \frac{1}{x^2} \right\rangle - \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle \left\langle \frac{|\nabla\psi|^2}{x^2} \right\rangle \right.$$
$$\left. + \left( \frac{2\pi g}{\chi'} \right)^2 \left[ \left\langle \frac{1}{x^2 |\nabla\psi|^2} \right\rangle - \left\langle \frac{1}{x^2} \right\rangle \left\langle \frac{1}{|\nabla\psi|^2} \right\rangle \right] \right\}.$$

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